

Quasi Cyclic LDPC Codes Based on Finite Set Systems

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Abstract

A finite set system (FSS) is a pair (V, \mathcal{B}) where V is a finite set whose members are called points, equipped with a finite collection of its subsets \mathcal{B} whose members are called blocks. In this paper, finite set systems are used to define a class of Quasi-cyclic low-density parity-check (LDPC) codes, called FSS codes, such that the constructed codes possess large girth and arbitrary column-weight distributions. Especially, the constructed column weight-2 FSS codes have higher rates than the column weight-2 geometric and cylinder-type codes with the same girths. To find the maximum girth of FSS codes based on (V, \mathcal{B}) , *inevitable walks* are defined in \mathcal{B} such that the maximum girth is determined by the smallest length of the inevitable walks in \mathcal{B} . Simulation results show that the constructed FSS codes have very good performance over the AWGN channel with iterative decoding and achieve significantly large coding gains compared to the random-like LDPC codes of the same lengths and rates.

Keywords: LDPC codes, Tanner graph, girth, closed walk.

1 Introduction

Low-density parity-check (LDPC) codes are forward error-correction codes, first proposed in 1962 by Gallager [1] and rediscovered [2] in 1996. The construction methods of LDPC codes can be divided into two categories: random-like methods [5]-[12] and mathematically structured methods [13]-[28]. Long random-like LDPC codes in general perform closer to the Shannon limit [4] than their equivalent structured LDPC codes; however, for practical lengths, well designed structured LDPC codes show better error correcting performance than the random-like ones. Quasi-cyclic (QC) LDPC codes are the most promising class of structured LDPC codes due to their ease of implementation and excellent performance over noisy channels when decoded by message-passing algorithms, such as sum-product algorithm [3], as extensive simulation studies have shown. An LDPC code is represented by a sparse parity-check matrix and its corresponding Tanner graph [14]. Tanner gave a lower bound on the minimum distance of a

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given LDPC code that grows exponentially with the girth of the Tanner graph representing the code.

PEG algorithm [31] is a graph conditioning technique to construct Tanner graphs of LDPC codes which possess large girths. The algorithm builds a Tanner graph by connecting the graphs nodes edge by edge provided that the added edge has minimal impact on the girth of the graph. Although PEG codes are among the best codes at short lengths [31], their major disadvantage is represented by their high implementation complexity that makes them impractical at very large lengths. On the other hand, structured LDPC codes, especially quasi-cyclic (QC) LDPC codes, such as FSS codes, have advantages over other types of LDPC codes in hardware implementation of encoding and decoding.

The overall bit-error rate (BER) and block-error rate (BLER) performance of an LDPC code is generally described by two different regions, i.e the waterfall region (WR) and the error-floor region (EFR). The WR corresponds to the low-to-medium signal-to-noise ratio (SNR) of the BER-SNR plot, and EFR [7] is located at the bottom of WR wherein the BER/BLER no longer exhibits the rapid improvement as in WR. While the performance in EFR mainly depends on the minimum-distance, stopping sets [10] and trapping sets [11], the girth influences the achievable BER/BLER in WR. On the other hand, the higher the girth, the faster the iteration-aided BER/BLER improvement, and this is why many construction techniques attempt to maximize the girth of the underlying graph [8]. Also, it has been shown that designing column-weight two LDPC codes [17, 18] with large girths, especially in the non-binary setting, is highly beneficial for the error floor performance.

Moura [21] defined a *structure graph* for column-weight two LDPC codes, or *cycle codes* [22], to construct regular structured LDPC codes with large girth. Asamov and Aydin [23] presented a greedy algorithm to construct regular cycle codes with arbitrarily girth. In [24], the authors have viewed a QC-LDPC code as a protograph code with circulant permutation matrices. They have also proposed a new combinatorial method for the construction of protographs whose protograph codes have girth larger than or equal to 14 and 18 for non applicable lengths. Also, a class of geometrically structured QC-LDPC codes with girth at most 18 [19] is constructed based on some algebraic tools, i.e. Steiner triple systems, integer lattices, and affine planes. Furthermore, a particular class of circulant LDPC codes referred to as cylinder-type LDPC (CT-LDPC) codes has been studied in [17] wherein for each $e \geq 2$, column-weight two CT-LDPC codes with girth $8e$ and rate $1/e$ have been constructed.

Despite having a suitable theoretical perspective for the mentioned constructions, it is desired to have efficient deterministic algorithms to produce LDPC codes of acceptable length, with flexible parameters such as rate, girth and row (column) weight distribution. In this paper, by introducing and applying a concept referred to as *inevitable walk*, we use the smallest length of such inevitable walks to give an efficient algorithm finding an upper-bound on the girth of QC-LDPC codes having a fixed mother matrix. Then, we introduce an efficient algorithm to produce mother matrices of large girths whose parity-check matrices have desirable column-weight distributions. As an advantage of this algorithm, the so constructed column-weight two LDPC codes have rates larger than the rates of the column-weight two geometric and cylinder-type codes with the same girths given in [17] and [20]. The basic mathematical structure employed in this paper is the class of finite set systems.

The outline of the paper is as follows. In section 2, we give some preliminaries and notations. To a circulant code, a graph, called *block-structure graph* (BSG) [20], is associated which is simpler than the graphs such as protograph [28] and voltage-graph [29]. By assigning a new matrix H to a given FSS, we define its corresponding FSS-codes as the QC-LDPC codes with mother matrix H . Construction of FSS-based LDPC codes is addressed in Section 4. By defining *inevitable walks* in a FSS, and by using BSG, we give a theorem which determines the maximum achievable girth by FSS-based codes derived from a given mother matrix. In section 4, we introduce two efficient algorithms to produce regular and non-regular FSS-based codes with desirable girths and column-weight distributions. Then, we propose two efficient algorithms producing regular FSS-based codes with arbitrary girths and column-weight distributions. Simulation results, given in Section 5, show that the new FSS-based codes can perform very well over the AWGN channel with iterative decoding, and they achieve a better coding gain compared to the random codes of the same lengths and rates.

2 Preliminaries and Notations

A *graph* G is a two-tuple consisting of a *vertex set* $V(G)$ and an *edge set* $E(G)$, where each element of $e \in E(G)$ is represented by two elements, not necessarily distinct, of $V(G)$ called the endpoints of this edge. A *bipartite graph* is a graph $G = (V, E)$ in which V can be divided into two disjoint sets A and B , such that every edge $e \in E$ connects one vertex in A and one in B . A *length- l walk* in G is a successive series of edges e_i and vertices v_j such as $v_1 e_1 v_2 e_2 \cdots v_l e_l v_{l+1}$, forming a continuous curve, i.e. each e_i connects v_i to v_{i+1} . A walk is closed if the initial and terminal vertices are the same, i.e. $v_1 = v_{l+1}$. A closed walk in which only the end vertices are the same is called a *cycle*, i.e. $v_i \neq v_j$ for each $1 \leq i < j \leq l + 1$ except for $i = 1, j = l + 1$. The *girth* of a graph is defined as the length of its shortest cycles.

To a given parity-check matrix H , a bipartite graph referred to as its *Tanner graph* (TG) is associated in which one set of vertices, called the check nodes, represent the set of rows of H , and the other set, called the bit nodes (or symbol nodes) represent the set of columns of H . A symbol node is adjacent with a check node if and only if the entry of H located in the corresponding row and column is nonzero.

Let m and s be two nonnegative integers with $0 \leq s \leq m - 1$. The $m \times m$ circulant permutation matrix \mathcal{I}_m^s is the matrix obtained from $m \times m$ identity matrix \mathcal{I}_m by shifting its rows s positions to the left. It is clear that $\mathcal{I}_m^0 = \mathcal{I}$. For simplicity, \mathcal{I}_m^s is denoted by \mathcal{I}^s when m is known. It is noticed that this definition of circulant permutation matrix seems to be different with [16], i.e. the row-shift is right-wise instead of left-wise, however it can be seen easily that the same conclusions can be obtained.

The definition of the block-structure graph given in [20] is reproduced as follows. Let m, b and γ be positive integers and $b < \gamma$. Let $\mathcal{H} = (H_{i,j})_{b \times \gamma}$, wherein each $H_{i,j}$ is either a $m \times m$ circulant permutation matrix or the $m \times m$ all-zero matrix. Considering \mathcal{H} as a $b \times \gamma$ matrix with $m \times m$ entries, we refer to \mathcal{H} as a matrix having b block-rows and γ block-columns. For simplicity, the matrix \mathcal{H} and the QC-LDPC code represented by \mathcal{H} as its parity-check matrix are referred to as m -circulant matrix and m -circulant code, respectively. The block-structure graph associated to \mathcal{H} , denoted by $\text{BSG}(\mathcal{H})$, is defined as follows.

Definition 1 Consider a set of vertices $V = \{v_1, v_2, \dots, v_b\}$, where v_i represents the i th block-row of \mathcal{H} . For each $i, j \in \{1, \dots, b\}$ and $k \in \{1, \dots, \gamma\}$, where $H_{i,k} = \mathcal{I}^{s_1}$ and $H_{j,k} = \mathcal{I}^{s_2}$ for some $0 \leq s_1, s_2 \leq m-1$, two vertices $v_i, v_j \in V$ are joined by two directed edges labeled with (k, s) , from v_i to v_j , and (k, s') , from v_j to v_i , wherein $s = -s' = s_2 - s_1 \pmod{m}$. For each edge of the so constructed graph G , the first and second component of its label (k, s) are referred to as the *column index* and the *shift* of that edge, respectively. The resulting graph G is called the *block-structure graph* of H , and is denoted by $\text{BSG}(\mathcal{H})$.

Definition 2 Consider a m -circulant matrix \mathcal{H} and its associated BSG G . A length- l closed walk in G is given by a sequence of vertices $v_{i_1}, v_{i_2}, \dots, v_{i_l}, v_{i_{l+1}}$, where $i_{l+1} = i_1$, with edges e_1, e_2, \dots, e_l such that for each $1 \leq j \leq l$, the edge e_j , labeled with (k_j, s_j) , connects the vertices v_{i_j} and $v_{i_{j+1}}$ and that the following conditions hold:

1. Each edge e_j in the sequence e_1, e_2, \dots, e_l is repeated at most m times;
2. For each $1 \leq j \leq l$, $k_j \neq k_{j+1}$, where $k_{l+1} := k_1$, i.e. the index columns of successive edges are different;
3. $\sum_{j=1}^l s_j \equiv 0 \pmod{m}$, i.e. the shift sum of the edges is zero modulo m .

For simplicity, we may show this length- l closed walk, or briefly closed l -walk, in G by the following chain:

$$v_{i_1} \xrightarrow{(k_1, s_1)} v_{i_2} \xrightarrow{(k_2, s_2)} v_{i_3} \quad \dots \quad v_{i_{l-1}} \xrightarrow{(k_{l-1}, s_{l-1})} v_{i_l} \xrightarrow{(k_l, s_l)} v_{i_1}.$$

Example 1 Consider the following 6-circulant matrix \mathcal{H} .

$$\mathcal{H} = \begin{pmatrix} \mathcal{I} & & \mathcal{I} & \mathcal{I} & \mathcal{I} \\ \mathcal{I}^1 & \mathcal{I} & & \mathcal{I}^4 & \\ & \mathcal{I}^3 & \mathcal{I}^2 & & \mathcal{I}^1 \\ & & & \mathcal{I}^5 & \mathcal{I} \end{pmatrix}$$

Figure 1 shows $\text{BSG}(\mathcal{H})$, wherein each two-tuple near an arrow, shows the label of that edge from its initial vertex to its terminal vertex. The following chain shows a closed 4-walk in $\text{BSG}(\mathcal{H})$ as $1 \neq 4 \neq 1 \neq 4$ and $1 + 2 + 1 + 2 = 0 \pmod{6}$.

$$v_1 \xrightarrow{(1,1)} v_2 \xrightarrow{(4,2)} v_1 \xrightarrow{(1,1)} v_2 \xrightarrow{(4,2)} v_1.$$

The relationship between the length of the shortest closed walks in BSG of a circulant code and the girth of its Tanner graph is considered in the following Lemma [20].

Lemma 1 For each circulant matrix \mathcal{H} , the girth of \mathcal{H} , i.e. the length of the shortest cycle in $\text{TG}(\mathcal{H})$, is twice of the length of the shortest closed walk in $\text{BSG}(\mathcal{H})$.

In the sequel, the definition of a (v, b, t) -finite set system (see [30]) and some examples are presented.

K	R	Λ_2	Λ_t	Other	Name
$[k, \dots, k]$	$[r, \dots, r]$	$\{\lambda_2\}$	$\{\lambda_t\}$		t – design
$[k, \dots, k]$	$[r, \dots, r]$	$\{\lambda_2\}$	$\{\lambda_t\}$	$\lambda_t = 1$	Steiner system
$[k, \dots, k]$	$[r, \dots, r]$	$\{\lambda\}$			balanced incomplete block design (BIBD)
	$[r, \dots, r]$	$\{\lambda\}$			(r, λ) – design
		$\{\lambda\}$			pairwise balanced design (PBD)
		$\{\lambda\}$		$\lambda = 1$	linear space
			$\{\lambda_t\}$		t – wise balanced design

Table 1: Some of the set systems that result when at least one of the balance sets $\{\Lambda_t\}$ is a singleton.

Definition 3 A (v, b, t) -finite set system, denoted by $\text{FSS}(v, b, t)$, is a pair (V, \mathcal{B}) where V is a finite set of size v whose members are called points, equipped with a finite collection \mathcal{B} of size b consisting subsets of V whose members are called blocks, together with a positive integer t less than or equal to the maximum block-size among the size of blocks in \mathcal{B} .

According to Definition 3, a block B in \mathcal{B} may be repeated, so, the word "collection" is preferred to "set". For a given $\text{FSS}(v, b, t)$, we show its points and collection of blocks by $V = \{1, 2, \dots, v\}$ and $\mathcal{B} = [B_1, B_2, \dots, B_b]$, respectively. For such a finite set system, we set the followings:

- $K = [|B_1|, |B_2|, \dots, |B_b|]$ is the collection of block sizes;
- For each $x \in V$, let r_x be the replication number of x in the blocks of \mathcal{B} , i.e. the number of blocks $B \in \mathcal{B}$ containing x and let $R = [r_x : x \in V]$ be the collection of replication numbers;
- For each $0 \leq i \leq t$ and each i -subset T of V , let $\Lambda_{i,T}$ be the number of blocks $B \in \mathcal{B}$ containing T and let $\Lambda_i = \{\Lambda_{i,T} : T \subseteq V, |T| = i\}$. Therefore, $\Lambda_0 = \{b\}$ and Λ_1 is the set of distinct elements in R .

For a given $\text{FSS}(v, b, t)$, a set of distinct points x_1, x_2, \dots, x_t in V are called *co-block* if there exists at least one block $B_i \in \mathcal{B}$ such that $x_1, \dots, x_t \in B$.

Example 2 The following 10 blocks define a $\text{FSS}(8, 10, 3)$: $B_1 = \{1, 2, 4, 5\}$, $B_2 = \{1, 2, 3, 7\}$, $B_3 = \{1, 3, 5, 8\}$, $B_4 = \{2, 3, 5, 6\}$, $B_5 = \{2, 3, 4, 8\}$, $B_6 = \{3, 4, 6, 7\}$, $B_7 = \{4, 5, 7, 8\}$, $B_8 = \{1, 4, 6, 8\}$, $B_9 = \{1, 5, 6, 7\}$, $B_{10} = \{2, 6, 7, 8\}$. For this, $K = [3, 3, \dots, 3]$, $R = [5, 5, \dots, 5]$, $\Lambda_0 = \{10\}$, $\Lambda_1 = \{5\}$, $\Lambda_2 = \{2\}$ and $\Lambda_3 = \{0, 1\}$.

Setting restrictions on $\{\Lambda_t\}$ s imposes specific structures on the system. Restriction to a single element balances the structure in some regard. Table 1 considers the effect of restricting these sets to singleton sets. Indeed, the set of so obtained systems are among the most central objects in design theory and have been studied extensively. If in a $\text{FSS}(v, b, t)$ we have $K = [k, \dots, k]$, $R = [r, \dots, r]$, $\Lambda_t = \{\lambda_t\}$, then the system is called t -design. Moreover, if $\lambda_t = 1$, then the t -design is called Steiner system $S(t, k, v)$. A *balanced incomplete block design* ($\text{BIBD}(v, k, \lambda)$) is a 2-design with $\lambda = \lambda_2$. For example, the FFS explained in example 2 with

bloks $\mathcal{B} = [B_1, B_2, \dots, B_{10}]$ can be considered as an BIBD(8, 4, 2). A *pairwise balanced design* (PBD) is a FSS(v, b, t) with $\Lambda_2 = \{\lambda\}$.

The incidence matrix of a given FSS is defined in the following.

Definition 4 Let (V, \mathcal{B}) be a FSS(v, b, t) where $\mathcal{B} = [B_1, B_2, \dots, B_b]$. Let $\{T_1, T_2, \dots, T_\gamma\}$ be the set of all $(t-1)$ -subsets of V such that for each T_i there are at least two distinct blocks containing T_i . We assign a $(0, 1)$ -matrix $H = (h_{i,j})_{b \times \gamma}$ to (V, \mathcal{B}) wherein the i -th row (j -th column) of H represents B_i (resp. T_j), and $h_{i,j} = 1$ if and only if $T_j \subseteq B_i$. The matrix H is referred to as the *incidence matrix* of (V, \mathcal{B}) .

For example, the incidence matrix H associated with the system defined in Example 2 is given by (1).

$$H = \begin{matrix} & \begin{matrix} 1,2 & 1,3 & 1,4 & 1,5 & 1,6 & 1,7 & 1,8 & 2,3 & 2,4 & 2,5 & 2,6 & 2,7 & 2,8 & 3,4 & 3,5 & 3,6 & 3,7 & 3,8 & 4,5 & 4,6 & 4,7 & 4,8 & 5,6 & 5,7 & 5,8 & 6,7 & 6,8 & 7,8 \end{matrix} \\ \begin{matrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \\ B_9 \\ B_{10} \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{matrix} \quad (1)$$

In definition 4, without loss of generality, we may assume that $t = 2$, as it is clear that the incidence matrix of a FSS(v, b, t) can be considered as the incidence matrix of a FSS($v', b, 2$), where $v' \leq \binom{v}{t-1}$. Hence in the rest of the paper, by FSS we mean a FSS($v, b, 2$) for some positive integers v and b . In this case, γ is equal to v .

Example 3 The incidence matrix of the FSS(8, 10, 3), represented by (1), can be considered as the base matrix of a FSS(28, 10, 2), (V, \mathcal{B}) , with $V = \{1, 2, \dots, 28\}$ and $\mathcal{B} = [B_1, B_2, \dots, B_{10}]$, for $B_1 = \{1, 3, 4, 9, 10, 19\}$, $B_2 = \{1, 2, 6, 8, 12, 17\}$, $B_3 = \{2, 4, 7, 15, 18, 25\}$, $B_4 = \{8, 10, 11, 15, 16, 23\}$, $B_5 = \{8, 9, 13, 14, 18, 22\}$, $B_6 = \{14, 16, 17, 20, 21, 26\}$, $B_7 = \{19, 21, 22, 24, 25, 28\}$, $B_8 = \{3, 5, 7, 20, 22, 27\}$, $B_9 = \{4, 5, 6, 23, 24, 26\}$, $B_{10} = \{11, 12, 13, 26, 27, 28\}$.

Definition 5 Let (V, \mathcal{B}) be a FSS($v, b, 2$) where $\mathcal{B} = [B_1, B_2, \dots, B_b]$ and let m be a positive integer. To the element $i \in B_j$ a non-negative integer $s_{i,j}$ is assigned. A finite sequence $S = (s_{i,j})_{i \in B_j, 1 \leq j \leq b}$ on \mathcal{B} is called a *shift sequence* of order m if each $s_{i,j}$ is in \mathbb{Z}_m .

Let $S = (s_{i,j})_{i \in B_j, 1 \leq j \leq b}$ be a shift sequence of order m on \mathcal{B} , and $\mathcal{H} = (H_{i,j})_{v \times b}$ be the matrix consisting of $m \times m$ block circulant matrices $H_{i,j}$, where:

$$H_{i,j} = \begin{cases} \mathcal{I}^{s_{i,j}} & i \in B_j, \\ 0 & i \notin B_j. \end{cases} \quad (2)$$

A finite set system code (FSS-code) based on \mathcal{B} and S , denoted by $C_{m,\mathcal{B},S}$, is defined as the quasi-cyclic LDPC code with the parity-check matrix \mathcal{H} if $b < \binom{v}{t-1}$ or its transpose \mathcal{H}^T , elsewhere. Hereinafter, without loss of generality, we assume that $b > \binom{v}{t-1}$.

Example 4 Let $V = \{1, 2, \dots, 6\}$ and $\mathcal{B} = [\{1, 2, 5\}, \{1, 2, 6\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 4\}, \{2, 3, 6\}, \{2, 4, 5\}, \{3, 5, 6\}, \{4, 5, 6\}]$. Then (V, \mathcal{B}) , known as BIBD(6,3,2), is a FSS(6, 10, 2).

Now let $S = (s_{i,j})_{i \in B_j, 1 \leq j \leq 10}$ be a shift sequence on (V, \mathcal{B}) and \mathcal{H} be the associated parity-check matrix. This code has rate at least $1 - 6/10 = 0.4$. Without loss of generality, up to code equivalence, we may assume that the first shift $s_{i,j}$ for each column of \mathcal{H} is zero, i.e. $s_{1,1} = s_{1,2} = s_{1,3} = s_{1,4} = s_{1,5} = s_{2,6} = s_{2,7} = s_{2,8} = s_{3,9} = s_{4,10} = 0$. Thus

$$\mathcal{H} = \begin{pmatrix} \mathcal{I} & \mathcal{I} & \mathcal{I} & \mathcal{I} & \mathcal{I} & 0 & 0 & 0 & 0 & 0 \\ \mathcal{I}^{s_{2,1}} & \mathcal{I}^{s_{2,2}} & 0 & 0 & 0 & \mathcal{I} & \mathcal{I} & \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{I}^{s_{3,3}} & \mathcal{I}^{s_{3,4}} & 0 & \mathcal{I}^{s_{3,6}} & \mathcal{I}^{s_{3,7}} & 0 & \mathcal{I} & 0 \\ 0 & 0 & \mathcal{I}^{s_{4,3}} & 0 & \mathcal{I}^{s_{4,5}} & \mathcal{I}^{s_{4,6}} & 0 & \mathcal{I}^{s_{4,8}} & 0 & \mathcal{I} \\ \mathcal{I}^{s_{5,1}} & 0 & 0 & \mathcal{I}^{s_{5,4}} & 0 & 0 & 0 & \mathcal{I}^{s_{5,8}} & \mathcal{I}^{s_{5,9}} & \mathcal{I}^{s_{5,10}} \\ 0 & \mathcal{I}^{s_{6,2}} & 0 & 0 & \mathcal{I}^{s_{6,5}} & 0 & \mathcal{I}^{s_{6,7}} & 0 & \mathcal{I}^{s_{6,9}} & \mathcal{I}^{s_{6,10}} \end{pmatrix}.$$

The numbers R and K in definition 3 are the row and column-weight distributions of the incidence matrix H , respectively. Hence, by choosing a FSS($v, b, 2$) with desired numbers R and K we can construct parity-check matrices having suitable column-weight and row-weight distributions. On the other hand, the rate of a FSS-code $C_{m,\mathcal{B},S}$ is at least $1 - \frac{\min\{b,v\}}{\max\{b,v\}}$. Hence, by increasing $|v - b|$, we hope that the obtained rate will be large enough. In order to have the girth of \mathcal{H} large enough, we first note that the girth of $C_{m,\mathcal{B},S}$ represented by \mathcal{H} is upper-bounded by $g(\mathcal{B})$, where $g(\mathcal{B})$ is the largest achievable girth for all possible integers m and shift-sequences S on m .

Let H be the incidence-matrix of (V, \mathcal{B}) . Using Proto-graphs [28], Kim et. al. [24] have introduced some incidence matrices, denoted P_{2i} , and have claimed that if H does not contain P_{2i} and P_{2i}^T for all $i < g$, then $g(\mathcal{B})$ will be at least $2g$. But, checking this condition, i.e. ensuring that P_{2i} and P_{2i}^T , $i < g$, are not as sub-matrices in H , especially when g is large, is a process with high computational complexity as H can include any row (column) permutation of P_{2i} or P_{2i}^T , as its sub-matrices.

In this paper, by introducing some special chains in (V, \mathcal{B}) , called inevitable walks, we propose a theorem that efficiently determines $g(\mathcal{B})$. Then, we propose two efficient algorithms for finding (V, \mathcal{B}) with a desired girth $g(\mathcal{B})$. Finally, we propose a modified version of the code-generating algorithm given in [19], such that for a given (V, \mathcal{B}) the algorithm efficiently generates FSS-codes with a desired girth. First, we define inevitable walks in a FSS as they have an important role in finding the maximum achievable girth by FSS-codes based on a given mother matrix.

Definition 6 Let $V = \{1, 2, \dots, v\}$ and $\mathcal{B} = [B_1, B_2, \dots, B_b]$ form a FSS($v, b, 2$). An inevitable walk of length ℓ , briefly ℓ -inevitable walk, in \mathcal{B} consists of a list of integers $i_1, i_2, \dots, i_\ell \in V$ and $k_1, k_2, \dots, k_\ell \in \{1, 2, \dots, b\}$, denoted by $[i_1, k_1, i_2, k_2, \dots, i_\ell, k_\ell]$, such that the following conditions hold wherein $i_{\ell+1} = i_1, k_{\ell+1} = k_1, i_{\ell+2} = i_2, k_{\ell+2} = k_2, \dots$:

1. $i_j \neq i_{j+1}, i_j, i_{j+1} \in B_{k_j}$ for $1 \leq j \leq \ell$.
2. For $1 \leq j \leq \ell$, if $k_j = k_{j+1}$ then $i_j \neq i_{j+2}$.

3. For some p , the vectors $(i_1, k_1, i_2), (i_2, k_2, i_3), \dots, (i_{\ell-1}, k_{\ell-1}, i_\ell), (i_\ell, k_\ell, i_1)$ can be partitioned into p sets of the form $A_j = \{(i_{j_1}, k_j, i_{j_2}), (i_{j_2}, k_j, i_{j_3}), \dots, (i_{j_t-1}, k_j, i_{j_t}), (i_{j_t}, k_j, i_{j_1})\}$, $1 \leq j \leq p$, where t depends on j .

Let G be the BSG of a parity-check matrix of a given $\text{FSS}(m, \mathcal{B}, S)$ code. If $[i_1, k_1, i_2, k_2, \dots, i_\ell, k_\ell]$ is an ℓ -inevitable walk in \mathcal{B} , then the following ℓ -closed walk always exists in G , regardless of the values of m and S .

$$v_{i_1} \xrightarrow{(k_1, s_1)} v_{i_2} \xrightarrow{(k_2, s_2)} v_{i_3} \dots v_{i_{\ell-1}} \xrightarrow{(k_{\ell-1}, s_{\ell-1})} v_{i_\ell} \xrightarrow{(k_\ell, s_\ell)} v_{i_1},$$

where $s_j = s_{i_{j+1}, k_j} - s_{i_j, k_j} \bmod m$ for $i_j, i_{j+1} \in B_{k_j}$.

Example 5 Let (V, \mathcal{B}) be the $\text{FSS}(6, 10, 2)$ given by example 4. The sequence $[1, 2, 2, 1, 5, 4, 1, 1, 2, 2, 1, 4, 5, 1]$ is a 7-inevitable walk in \mathcal{B} . In fact, the vectors $(1, 2, 2), (2, 1, 5), (5, 4, 1), (1, 1, 2), (2, 2, 1), (1, 4, 5)$, and $(5, 1, 1)$ can be partitioned to the sets $A_1 = \{(1, 2, 2), (2, 2, 1)\}$, $A_2 = \{(1, 1, 2), (2, 1, 5), (5, 1, 1)\}$, and $A_3 = \{(5, 4, 1), (1, 4, 5)\}$. This inevitable walk is represented by the following chain:

$$v_1 \xrightarrow{(2, s_1)} v_2 \xrightarrow{(1, s_2)} v_5 \xrightarrow{(4, s_3)} v_1 \xrightarrow{(1, s_4)} v_2 \xrightarrow{(2, s'_1)} v_1 \xrightarrow{(4, s'_3)} v_5 \xrightarrow{(1, s_5)} v_1,$$

where $s_1 = -s'_1 = s_{2,2} - s_{1,2}$, $s_2 = s_{5,1} - s_{2,1}$, $s_3 = -s'_3 = s_{1,4} - s_{5,4}$, $s_4 = s_{2,1} - s_{1,1}$ and $s_5 = s_{1,1} - s_{5,1}$. It is clear that the sum of the shift of consecutive edges of this chain, shown by a 7-inevitable walk in Figure 2, is zero modulus m , for any m , i.e.

$$\begin{aligned} s_1 + s_2 + s_3 + s_4 + s'_1 + s'_3 + s_5 = \\ (s_{2,2} - s_{1,2}) + (s_{5,1} - s_{2,1}) + (s_{1,4} - s_{5,4}) + (s_{2,1} - s_{1,1}) \\ + (s_{1,2} - s_{2,2}) + (s_{5,4} - s_{1,4}) + (s_{1,1} - s_{5,1}) = 0 \pmod{m}. \end{aligned}$$

Let (V, \mathcal{B}) be a $\text{FSS}(v, b, 2)$ and $H = (H_{i,j})_{b \times v}$ be its associated incidence matrix. Then, an ℓ -inevitable walk in \mathcal{B} is a closed walk that exists in the BSG of $\text{FSS}(m, \mathcal{B}, S)$ codes regardless of the values of m and S .

The upper-bound $g(\mathcal{B})$ is obtained from the shortest inevitable walks in \mathcal{B} by the following theorem.

Theorem 1 The number $g(\mathcal{B})$ is the smallest integer 2ℓ , such that there is at least one ℓ -inevitable walk in \mathcal{B} .

Proof. We should show that for a given (V, \mathcal{B}) , the maximum achievable girth of $\text{FSS}(m, \mathcal{B}, S)$ codes, for all m and S , is the smallest 2ℓ , such that there is an ℓ -inevitable walk in \mathcal{B} . Let $W = [i_1, k_1, \dots, i_\ell, k_\ell]$ be an inevitable walk in \mathcal{B} and that ℓ is the smallest such number. We show that the girth of any $\text{FSS}(m, \mathcal{B}, S)$ code is upper-bounded by 2ℓ . If \mathcal{H} is the parity-check matrix of a $\text{FSS}(m, \mathcal{B}, S)$ code, for some m and S , and G is the BSG(\mathcal{H}), then regardless of the values of m and S , G contains the ℓ -closed walk represented by the following chain.

$$v_{i_1} \xrightarrow{(k_1, s_1)} v_{i_2} \xrightarrow{(k_2, s_2)} v_{i_2} \dots v_{i_\ell} \xrightarrow{(k_\ell, s_\ell)} v_{i_1}, \quad (3)$$

where $s_j = s_{i_{j+1}, k_j} - s_{i_j, k_j} \bmod m$, $1 \leq j \leq \ell$. As \mathcal{C} is an inevitable walk, the vectors $(i_1, k_1, i_2), (i_2, k_2, i_3), \dots, (i_\ell, k_\ell, i_1)$ can be partitioned to the p sets $A_j = \{(i_{j_1}, k_j, i_{j_2}), (i_{j_2}, k_j, i_{j_3}), \dots,$

$(i_{j_t}, k_j, i_{j_1})\}$, $1 \leq j \leq p$, where $t = |A_j|$ and the points $i_{j_1}, i_{j_2}, \dots, i_{j_t}$ are B_{k_j} co-block. Hence, $\sum_{h=1}^{|A_j|} (s_{i_{j_{h+1}}, k_j} - s_{i_{j_h}, k_j}) \equiv 0 \pmod{m}$ for any $1 \leq j \leq p$, and $\sum_{i=1}^\ell s_i = \sum_{j=1}^p \sum_{h=1}^{|A_j|} (s_{i_{j_{h+1}}, k_j} - s_{i_{j_h}, k_j}) \equiv 0 \pmod{m}$ where $i_{j_{|A_j|+1}} = i_{j_1}$.

Conversely, if the ℓ -closed walk, represented by the chain given by (3), is in G , regardless of the values of m and S , then it is easily verified that each edge of this walk is a part of an ℓ -closed walk in G , between some points $i_{j_1}, i_{j_2}, \dots, i_{j_t}$ belonging to a block B_{k_j} , for some $1 \leq j \leq p$. Now, let $A_j = \{(i_{j_1}, k_j, i_{j_2}), (i_{j_2}, k_j, i_{j_3}), \dots, (i_{j_t}, k_j, i_{j_1})\}$. It is clear that $W = [i_1, k_1, \dots, i_\ell, k_\ell]$ is an ℓ -inevitable walk in \mathcal{B} . \square

3 Construction of FSS Codes with Arbitrary Girth

By Theorem 1, the number $g(\mathcal{B})$ is the maximum achievable girth by $\text{FSS}(m, \mathcal{B}, S)$ codes, for all m and S , and it is equal to 2ℓ , where ℓ is the smallest number such that \mathcal{B} contains an ℓ -inevitable walk. Thus, in order to construct FSS codes with large girths, we first need to design \mathcal{B} having $g(\mathcal{B})$ large enough, and then choose m and S properly, such that the associated $\text{FSS}(m, \mathcal{B}, S)$ code has the desired girth not greater than $g(\mathcal{B})$.

We introduce two structured methods to construct (V, \mathcal{B}) with desired $g(\mathcal{B})$. The first approach uses a recursive method to convert a FSS (V', \mathcal{B}') with $g(\mathcal{B}') = 2g$ to a new FSS (V, \mathcal{B}) with $g(\mathcal{B}) \geq 6g$. For given positive integers b and v and a collection of integers K , the second approach produces (V, \mathcal{B}) , if it exists, such that its incidence matrix $H = (h_{i,j})_{v \times b}$ has column weight distribution K and satisfies $g(\mathcal{B}) \geq 2g$.

3.1 Method 1

Let (V', \mathcal{B}') be a FSS with $g(\mathcal{B}') = 2g'$ and \mathcal{H} be the parity-check matrix of a $\text{FSS}(m, \mathcal{B}', S)$ code, for some m and S . If we consider \mathcal{H} as the incidence matrix of a new FSS, (V, \mathcal{B}) , then Theorem 2 of [24] guarantees that $g(\mathcal{B}) \geq 6g'$. Hence, to design (V, \mathcal{B}) with $g(\mathcal{B}) \geq 2g$, we use the following algorithm.

Algorithm I

1. Design (V', \mathcal{B}') , such that $3g(\mathcal{B}') \geq 2g$.
2. Choose a positive integer m and a shift-sequence S on m properly, such that their corresponding $\text{FSS}(m, \mathcal{B}', S)$ code, with the parity check matrix \mathcal{H} , has girth $2g' (\leq g(\mathcal{B}'))$, where $3g' \geq g$.
3. Consider \mathcal{H} as the incidence matrix of a new FSS (V, \mathcal{B}) where $g(\mathcal{B}) \geq 6g' \geq 2g$.

By continuing this process, one can obtain FSS codes with arbitrary large girths. Based on Algorithm I, Table 3 presents some new (V, \mathcal{B}) with $g(\mathcal{B}) = 2g$, derived from the initial (V', \mathcal{B}') with $g(\mathcal{B}') = 2g'$ where $g' = \lceil g/3 \rceil$.

Remark 1 We should note that the row (column) weight distribution of the incidence matrices of (V, \mathcal{B}) and (V', \mathcal{B}') in Method 1 are the same. Now, in order to construct (V, \mathcal{B}) with a incidence matrix having a desired column-weight distribution, we use the following construction method.

3.2 Method 2

Let b, g and v be some positive integers and $K = [k_1, \dots, k_b]$ be a collection of not necessarily distinct positive integers k_i . In the sequel, we provide an algorithm which inductively generates proper FSS($v, b, 2$) whose incidence matrices can be considered as the incidence matrices of some QC LDPC codes with column-weight distribution K and girth at least $2g$. Moreover, we assume $k_i > 1$ to avoid having column-weight 1 in the associated parity-check matrix \mathcal{H} . In order to construct a (V, \mathcal{B}) having incidence matrix with column-weight distribution K , we propose the following deterministic algorithm which inductively finds $\mathcal{B} = [B_1, B_2, \dots, B_b]$, $|B_i| = k_i$, $B_i \subseteq V = \{1, \dots, v\}$ with $g(\mathcal{B}) \geq 2g$.

An overview of the algorithm is as follows. Given a column-weight distribution $K = [k_1, \dots, k_b]$ and girth $2g$, we form the elements of the required \mathcal{B} sequentially from the elements of B_1 to B_b . In step e , assuming that the blocks $[B_1, \dots, B_{b'-1}, B'_{b'}]$, $b' \leq b$ and $B'_{b'} \subseteq B_{b'}$ are chosen, we determine A_{e+1} consisting of all elements of $\beta \in V$, such that for any element $\beta_{b',j} \in B'_{b'}$, the inevitable walks containing the edge $(\beta_{b',j}, \beta)$ in $\mathcal{B}_{e+1} = [B_1, \dots, B'_{b'-1}, B'_{b'} \cup \{\beta\}]$ have length at least $2g$, i.e. $g((\beta_{b',j}, \beta), \mathcal{B}_{e+1}) \geq 2g$. Also, in order to avoid checking edges repeatedly, we have assumed that $\beta > \max B'_{b'}$. If $A_{e+1} = \emptyset$, then we move one step back and replace A_e with $A_e - \{\beta_{b',|B'_{b'}|}\}$, choose an arbitrary element of the updated set A_e as $\beta_{b',|B'_{b'}|+1}$, update $B'_{b'} = B'_{b'} \cup \{\beta_{b',|B'_{b'}|+1}\}$ and $\mathcal{B}_e = [B_1, \dots, B'_{b'}]$, then determine A_{e+1} . If this process ends up with $A_1 = \emptyset$ we conclude that there is no (V, \mathcal{B}) with $g(\mathcal{B}) = 2g$. The formal step-by-step framework of the algorithm is as follows.

Algorithm II

- s1. Set $e = 1$, $V = \{1, 2, \dots, v\}$, $b' = 1$ and $A_1 := V$.
- s2. First choose $\beta_{1,1} \in A_1$ and set $B_1 = \{\beta_{1,1}\}$ and $\mathcal{B}_1 = [B_1]$.
- s3. In step e , assume that $\mathcal{B}_e = [B_1, \dots, B'_{b'}]$, $1 \leq b' \leq b$, has been constructed, such that $|B_i| = k_i$, for each $i < b'$, and $|B'_{b'}| \leq k_{b'}$.
- s4. In step $e+1$, if $|B'_{b'}| < k_{b'}$, then let A_{e+1} be the set of all $\beta \in V$, such that $\beta > \max B'_{b'}$ and $g((\beta_{b',j}, \beta), \mathcal{B}_{e+1}) \geq 2g$, for all $j \leq |B'_{b'}|$ where $\mathcal{B}_{e+1} = [B_1, \dots, B'_{b'-1}, B'_{b'} \cup \{\beta\}]$, otherwise if $|B'_{b'}| = k_{b'}$ then define $A_{e+1} := V$, replace b' with $b' + 1$ ($b' \rightarrow b' + 1$), choose $\beta \in V$ and set $B'_{b'} = \{\beta\}$, and go to s6.
- s5. If $A_{e+1} = \emptyset$, then $A_e \rightarrow A_e - \{\beta_{b',|B'_{b'}|}\}$, $e \rightarrow e - 1$.
- s6. If $e < 0$, then there is no solution, so go to end.
- s7. If $A_{e+1} \neq \emptyset$, then choose $\beta_{b',|B'_{b'}|+1} \in A_{e+1}$, and update $B'_{b'} = B'_{b'} \cup \{\beta_{b',|B'_{b'}|+1}\}$, $\mathcal{B}_{e+1} = [B_1, \dots, B'_{b'}]$ and $e \rightarrow e + 1$; else go to s5.
- s8. If $e < \sum_{i=1}^b k_i$, then go to s4; else print $\mathcal{B} = \mathcal{B}_e$ as a solution and go to end.

For simplicity of the notations, we considered the convention that $B_i = \{\beta_{i,1}, \dots, \beta_{i,k_i}\}$, $1 \leq i \leq b$, and that $\mathcal{B}_e = [B_1, \dots, B_{b'-1}, B'_{b'}]$ ($b' \leq b$ and $B'_{b'} \subseteq B_{b'}$) is the sub-FSS of \mathcal{B} , constructed in the first e stages of the algorithm. In fact, $\mathcal{B}_1 = [\{\beta_{1,1}\}]$, $\mathcal{B}_2 = [\{\beta_{1,1}, \beta_{1,2}\}]$,

$\dots, \mathcal{B}_{k_1} = [B_1], \mathcal{B}_{k_1+1} = [B_1, \{\beta_{2,1}\}], \mathcal{B}_{k_1+2} = [B_1, \{\beta_{2,1}, \beta_{2,2}\}], \dots, \mathcal{B}_{k_1+k_2} = [B_1, B_2], \dots, \mathcal{B}_{k_1+\dots+k_b} = \mathcal{B}$. Also, $g((i_1, i_2), \mathcal{B}_e)$, for some $i_1 \in B'_b$ and $i_2 \in V$, is the smallest integer number 2ℓ , such that there is an ℓ -inevitable walk $[i_1, j_1, i_2, j_2, \dots, i_\ell, j_\ell]$ in \mathcal{B}_{e+1} , where $\mathcal{B}_{e+1} = [B_1, \dots, B'_b \cup \{i_2\}]$ if $|B'_b| < |B_b|$, otherwise $\mathcal{B}_{e+1} = [B_1, \dots, B_b, \{i_2\}]$. If there is no such ℓ -inevitable walk in \mathcal{B}_{e+1} , we set $g((i_1, i_2), \mathcal{B}_e) = 2g$ as condition $g(\mathcal{B}) \geq 2g$ must be satisfied. Note that the maximum element in $B \subseteq V$ is denoted by $\max B$.

Applying algorithm II, we have constructed a set of (V, \mathcal{B}) with $14 \leq g(\mathcal{B}) \leq 20$ and rates $r = 1 - \frac{v}{b}$, with parity-check matrices having column-weights varying among 2, 3 and 4 (see Table 4). Moreover, as an advantage of algorithm II, Table 5 presents some (V, \mathcal{B}) which have column-weight two mother matrices, satisfy $g(\mathcal{B}) \geq 24$, and have rates r better than the rates found in [17] and [19]. Also, the algorithms outputs show that the rates of such column-weight two FSS codes grow with the row-number of their mother matrices.

3.3 Construction of FSS Codes

Given a positive integer g , let (V, \mathcal{B}) be a FSS with $g(\mathcal{B}) = 2g$, where $\mathcal{B} = [B_1, \dots, B_b]$. Here, we provide a deterministic algorithm that generates FSS codes with girth $2g' (\leq 2g)$. This algorithm is a modified version of the code-generating algorithm given in [19]. To determine the shift sequence $S = (s_{i,j})_{i \in B_j, 1 \leq j \leq b}$ on \mathcal{B} , for a sufficiently large block size m , initially we consider S as the length- $\sum_{i=1}^b k_i$ vector $S := [s_{1,1}, \dots, s_{k_1,1}, s_{1,2}, \dots, s_{k_2,2}, \dots, s_{1,b}, \dots, s_{k_b,b}]$, where $k_i = |B_i|$. Assuming that the first e elements of S , denoted by $S_e := [s_1, s_2, \dots, s_e]$, are chosen, we determine A_{e+1} consisting of all elements of V such that for any element $s \in A_{e+1}$ the FSS($m, \mathcal{B}_{e+1}, S_{e+1} = [s_1, s_2, \dots, s_e, s_{e+1} := s]$) code has girth at least $2g'$. If $A_{e+1} = \emptyset$, then we move one step back and replace A_e with $A_e - \{s_e\}$. If this process ends up with $A_1 = \emptyset$, we conclude that for the given m and (V, \mathcal{B}) , there is no FSS(m, \mathcal{B}, S) code with girth $2g'$. This algorithm efficiently generates FSS codes with girth at most 20. A list of so constructed codes are given in Table 2. In the following, we find the complexity of the proposed algorithm generating FSS codes with girth g .

3.4 Complexity of the Algorithm

Let s_1, s_2, \dots, s_e be chosen suitably and r_{\max}^e be the maximum number of occurrence of each element of $V = \{1, 2, \dots, v\}$ in sub-blocks of \mathcal{B} induced in step e , denoted by \mathcal{B}_e , and k_{\max}^e be the maximum size of blocks of \mathcal{B}_e . Then, we must find all elements $0 \leq s \leq m-1$ and all chains $i_0 \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_{l-1} \rightarrow i_l = i_0$, each i_j, i_{j+1} is co-block in \mathcal{B}_e , such that all equations $\sum_{k=0}^l s_{i_k, i_{k+1}} (1 \leq l \leq \frac{g}{2} - 1, s_{i_0, i_1} = s)$ are not equal to zero in modulus of m . In step e of the algorithm, two vertices i_0 and i_1 in the chains $i_0 \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_{l-1} \rightarrow i_l = i_0$ are given, so we must find all possible indices i_2, \dots, i_{l-1} to check whether the equations $\sum_{k=0}^l s_{i_k, i_{k+1}}$ are equal to zero or not. However, for each $2 \leq j \leq l-1$, the number of choices i_j is at most $(r_{\max}^e - 1)(k_{\max}^e - 1)$, because for each $1 \leq j \leq l$, i_j and i_{j-1} is co-block and i_j and i_{j-2} is not co-block; hence if i_{j-1} and i_{j-2} are given from block B , then i_j is one of co-block elements of i_{j-1} , except block B , whereas i_{j-1} is repeated at

most $r_{\max}^e - 1$ in other blocks. Therefore, all possible vectors (i_2, \dots, i_{l-1}) are enumerated at most $(r_{\max}^e - 1)^{l-2}(k_{\max}^e - 1)^{l-2}$, $l \leq \frac{g}{2} - 1$. But $0 \leq s \leq m - 1$. So, the complexity of step e of the algorithm is $O((r_{\max}^e - 1)^{\frac{g}{2}-3}(k_{\max}^e - 1)^{\frac{g}{2}-3}m^e)$ and the overall complexity is about $\sum_{e=1}^{(k-1)b} O(r_{\max}^e - 1)^{\frac{g}{2}-3}(k_{\max}^e - 1)^{\frac{g}{2}-3}m^e = O(b(k-1)^{\frac{g}{2}-2}(r-1)^{\frac{g}{2}-3}m^{b(k-1)})$, where $b = |\mathcal{B}|$, $r = \max r_{\max}^e$ and $k = \max k_{\max}^e$. For example, to find appropriate shift-sequence S in Example 4 with $g(\mathcal{H}) = 14$, the complexity is $O(10 \times 2^5 \times 4^4 \times m^{20}) = O(m^{20})$.

Therefore, the algorithm has a polynomial complexity with respect to the input value m , if b , k , g and r are fixed. On the other hand, if g and r are fixed, the complexity grows up exponentially with respect to k or b . In this case, the exhaustive search is impractical and a heuristic method can be used to speed up the process of finding a satisfactory solution via selecting the shifts in each step randomly.

4 Simulation Results

This section provides a bit-error rate (BER) performance comparison over the AWGN channel with BPSK modulation between some FSS codes with different girths, on one hand, and random-like counterparts [2], PEG codes [31], cylinder-type LDPC codes [20] and QC LDPC codes based on Steiner triple systems STS(9), STS(13) and the configuration Aff*(16) [19] with the same rates and lengths, on the other hand. The binary FSS codes have been decoded with the iterative sum-product algorithm (SPA) and all randomly generated codes given in this paper are constructed using Radford Neal's software [32], and they have as few number of 4-cycles as possible.

In the performance figures, the notation FSS($\lambda(x); gb$), $c \in \{2, 3\}$, refers to a girth b -FSS code with column weight distribution $\lambda(x)$. In addition, PEG($n; gb$) is used to denote the girth- b LDPC code of length n constructed by PEG and QPEG($x \times y; gb$) is used to denote the girth- b QC LDPC code obtained by applying the proposed algorithm in Section 3.3 to a $x \times y$ base matrix generated by PEG. Also, Rand(n, cb) denotes a random-like column-weight b code of length n constructed with Neal's software. Moreover, by C($m; gb$), STS13($m; gb$) and con12($m; gb$), we mean the cylinder-type QC LDPC codes [20] and QC LDPC codes based on STS(13) and Aff*(16) [19], respectively.

Although the high-length random codes usually have better performance rather than the structured ones, Figure 3 shows that the FSS codes with girth 20 and 18 constructed by Method I outperform the QPEG and Random-like codes. Moreover, the constructed FSS codes have remarkably coding gains, about 0.6 dB, over than cylinder-type QC LDPC codes with girths 10 and 12. Figure 4 shows that the FSS codes with girths 12 and 14 constructed by Method II outperform the QPEG codes with girth 12 and 14, respectively. Also, the constructed FSS codes have coding gains over than random-like counterparts and QC LDPC codes with girths 12 and 14 based on STS(13). Figure 5 highlights the positive role of girth in improving the efficiency of performance of column-weight two FSS codes. Moreover, column-weight two FSS codes with girth at least 32 outperform PEG and random-like LDPC codes. Finally, Figure 6 implies that the FSS codes with girths 14 and 16 outperform the random-like, QPEG, cylinder-type and Aff*(16) based QC LDPC codes.

The performance of an LDPC code depends not only on the girth of its Tanner graph,

but also on the structure of the Tanner graph and the weight distribution of the parity-check matrix. Although PEG algorithm is one of the best powerful algorithm to generate LDPC codes with large girth and good weight distribution in short-block-length, simulation results confirm that the constructed codes with large girth outperform than PEG codes in medium lengths. Maybe it is because the real girth of the codes constructed by PEG algorithm is less than their target girth in medium lengths when the target girth is high. Moreover, in constructing FSS codes with large girth, some attempts have been made to construct base matrices with good weight distribution profiles. Therefore, FSS codes with large girth outperform than PEG and QPEG codes. Besides, there is considerable complexity to generate high girth PEG codes when the length of the code enlarges. PEG codes are not among the QC LDPC codes, so their encoding and decoding process has more complexity rather than FSS codes. QPEG is a good approach to construct QC LDPC codes having mother matrices constructed by PEG algorithm. Nevertheless, the weigh distribution of QPEG is not optimal as well as PEG codes.

5 Conclusion

In this paper, we proposed some binary well-structured LDPC codes based on some combinatorial designs, called finite set systems. The girth and column-weight distribution of parity-check matrices of these proposed codes are arbitrary. Also, for a given binary matrix H , we determined the maximum achievable girth of QC-LDPC codes having mother matrix H . Then, we proposed two structured methods to construct mother matrices such that their associated QC-LDPC codes have large girths. Our constructed QC-LDPC codes with column-weight two have larger rates compared to the rate of geometric and cylinder-type LDPC codes with the same girth. Simulation results confirm that the constructed codes with larger girths perform better than the codes with smaller girths, and randomly constructed LDPC codes.

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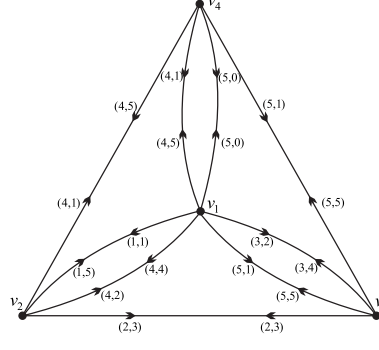


Figure 1: The BSG of the 6-circulant code given in Example 1.

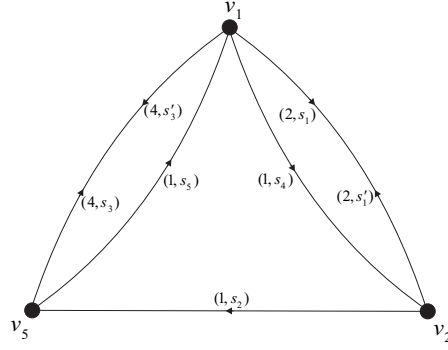


Figure 2: The length-7 closed walk associated with the length-7 inevitable walk in BSG of Example 5.

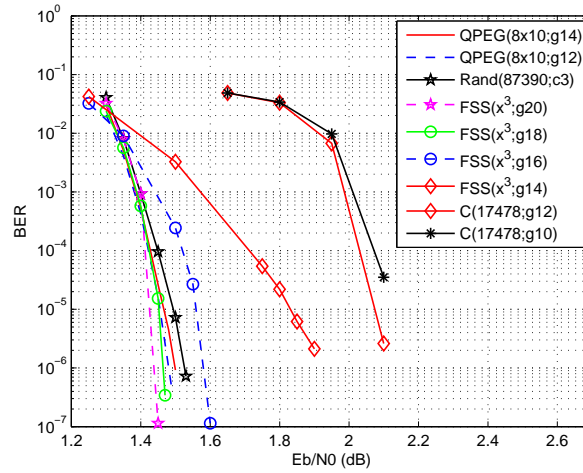


Figure 3: A comparison between some FSS codes constructed by method I, with different girths, random-like, QPEG and cylinder-type QC LDPC codes of length 87390 and rate about 0.2.

Table 2: Some $\text{FSS}(m, \mathcal{B}, S)$ codes with girth $2g$ and length n derived from $\text{FSS}(v, b, 2)$

(v, b)	\mathcal{B}	\mathcal{R}	m	$2g$	n	S
(18, 30)	$\{ \{1, 2, 3\}, \{3, 11\}, \{2, 14\}, \{8, 17\}, \{10, 11\}, \{11, 16\}, \{10, 15, 16\}, \{7, 17\}, \{7, 16\}, \{15, 17\}, \{12, 14, 18\}, \{5, 6, 9, 18\}, \{14, 17\}, \{4, 6\}, \{6, 10, 14\}, \{4, 11, 17, 18\}, \{2, 4, 12\}, \{2, 10\}, \{4, 13\}, \{16, 17\}, \{8, 10\}, \{3, 9, 10\}, \{5, 8, 12, 16\}, \{7, 9, 13\}, \{4, 7\}, \{3, 7\}, \{1, 9, 15\}, \{8, 9\}, \{2, 7, 8\}, \{1, 18\} \}$	0.4	3	8	90	$[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 2, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 2, 2, 1, 0, 1, 1, 2, 0]$
			8	10	240	$[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 3, 4, 0, 2, 2, 0, 3, 1, 0, 1, 1, 5, 0, 2, 0, 4, 5, 3, 5, 3, 2, 7, 1]$
			10	12	300	$[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 5, 6, 1, 2, 8, 0, 5, 3, 3, 7, 3, 9, 12, 2, 3, 10, 12, 5, 18, 18, 17, 23, 15]$
			100	14	3000	$[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 3, 2, 6, 10, 1, 2, 7, 0, 10, 2, 6, 18, 1, 16, 25, 3, 10, 27, 46, 5, 33, 41, 58, 81, 52]$
			359	16	10770	$[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 3, 0, 0, 0, 0, 0, 0, 1, 0, 1, 5, 7, 0, 13, 4, 32, 22, 0, 20, 16, 22, 47, 3, 62, 92, 12, 22, 53, 88, 7, 76, 113, 131, 198, 115]$
			4000	18	120000	$[0, 3447, 3447, 3447, 3447, 3447, 3447, 3447, 3447, 3447, 3447, 3447, 3447, 3447, 270, 3447, 2793, 1875, 582, 228, 3573, 3224, 2570, 786, 1567, 1567, 1218, 2779, 446, 2417, 2750, 3743, 930, 2917, 3341, 3591, 1099, 3650, 1605, 1709, 2140, 1453]$
			40000	20	1200000	$[0, 8301, 19064, 19064, 19064, 19064, 19064, 19064, 19064, 19064, 19064, 19064, 19064, 5196, 19064, 19064, 19064, 19064, 29828, 29828, 26036, 24920, 8075, 11540, 15726, 27873, 28060, 15627, 15627, 11342, 8003, 22006, 18470, 15132, 1455, 7042, 9361, 7394, 11468, 4501, 20308, 8990, 1284, 26684, 15693]$
(14, 27)	$\{ \{1, 2\}, \{2, 7\}, \{5, 13\}, \{1, 3\}, \{9, 14\}, \{2, 8, 12, 13\}, \{1, 13\}, \{9, 14\}, \{7, 8\}, \{13, 14\}, \{4, 12\}, \{5, 6\}, \{5, 10\}, \{5, 10\}, \{6, 12\}, \{2, 3, 4, 6\}, \{4, 5\}, \{6, 8\}, \{1, 14\}, \{2, 10\}, \{10, 12\}, \{3, 7, 12, 14\}, \{4, 14\}, \{4, 8\}, \{5, 11\}, \{3, 11\}, \{9, 10\} \}$	0.48	4	8	108	$[0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 2, 1, 2, 0, 0, 0, 1, 0, 1, 0, 2, 0, 2, 0, 0, 0]$
			7	10	189	$[0, 5, 1, 1, 1, 1, 1, 1, 5, 5, 1, 1, 5, 2, 5, 2, 2, 2, 4, 1, 0, 4, 5, 0, 5, 0, 2, 1, 0, 3, 0, 0, 0]$
			15	12	405	$[0, 6, 4, 4, 4, 4, 4, 4, 11, 11, 4, 4, 4, 11, 12, 4, 4, 11, 5, 2, 11, 11, 2, 6, 2, 7, 6, 7, 7, 3, 4, 3]$
			100	14	2700	$[0, 16, 16, 16, 16, 16, 16, 16, 91, 91, 16, 16, 16, 16, 98, 32, 98, 98, 32, 53, 98, 35, 35, 71, 35, 35, 64, 74, 18, 60, 79, 94, 39]$
			175	16	4725	$[0, 126, 126, 126, 126, 126, 126, 126, 161, 168, 20, 20, 20, 20, 20, 21, 20, 20, 21, 82, 142, 116, 158, 130, 150, 123, 31, 59, 87, 107, 93, 150, 118]$
			700	18	18900	$[0, 511, 397, 397, 397, 397, 397, 397, 104, 626, 684, 684, 570, 570, 570, 565, 570, 570, 321, 122, 258, 206, 394, 273, 219, 567, 506, 36, 122, 57, 415, 518, 23]$
(15, 36)	$\{ \{1, 2\}, \{14, 15\}, \{6, 13, 14\}, \{8, 13\}, \{3, 6\}, \{2, 4\}, \{1, 4, 15\}, \{10, 12\}, \{5, 9\}, \{1, 9, 10\}, \{8, 14\}, \{11, 14\}, \{5, 7, 10\}, \{13, 15\}, \{4, 7, 8\}, \{5, 12, 14\}, \{5, 15\}, \{2, 12, 13\}, \{4, 13\}, \{11, 12\}, \{10, 13\}, \{6, 10, 11\}, \{2, 8\}, \{3, 7, 11\}, \{8, 15\}, \{8, 11\}, \{1, 3, 14\}, \{3, 5\}, \{1, 12\}, \{4, 5, 6\}, \{2, 6\}, \{3, 9\}, \{8, 10\}, \{2, 9\}, \{9, 11, 15\}, \{1, 7\} \}$	0.58	4	8	144	$[0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 2, 0, 0, 1, 2, 0, 1, 2, 0, 1, 0, 1, 2, 0, 1, 0, 2, 0, 1, 0]$
			13	10	468	$[0, 8, 8, 8, 8, 8, 8, 6, 8, 8, 8, 6, 8, 8, 6, 8, 8, 8, 6, 8, 8, 8, 1, 6, 1, 0, 6, 1, 1, 6, 5, 7, 11, 4, 5, 0, 5, 4, 7, 8, 10, 2, 11, 10, 1, 11, 3, 0]$
			40	12	1440	$[0, 21, 14, 14, 14, 14, 14, 30, 14, 14, 14, 14, 30, 14, 14, 30, 14, 14, 30, 14, 14, 14, 14, 14, 30, 5, 30, 5, 18, 18, 13, 18, 28, 22, 22, 13, 15, 23, 36, 37, 23, 19, 11, 32, 8, 14, 29, 3, 3, 13, 2]$
			250	14	9000	$[0, 126, 126, 126, 126, 126, 126, 161, 126, 126, 126, 126, 161, 126, 126, 161, 2, 126, 126, 215, 215, 215, 85, 85, 132, 55, 55, 109, 14, 104, 63, 152, 57, 201, 166, 28, 28, 221, 245, 199, 25, 184, 212, 119, 225, 118, 108, 15]$
			1000	16	36000	$[0, 948, 785, 785, 785, 785, 785, 207, 785, 785, 785, 785, 207, 785, 785, 495, 785, 785, 785, 144, 901, 144, 144, 980, 980, 691, 534, 787, 88, 283, 465, 228, 945, 889, 125, 193, 242, 732, 538, 670, 790, 922, 478, 656, 901, 606, 720, 22]$
			2000	18	72000	$[0, 12965, 12965, 12965, 12965, 12965, 9692, 11633, 9692, 9692, 9692, 9692, 9692, 9692, 16362, 16362, 9692, 9692, 16868, 5527, 16868, 773, 7948, 4676, 19659, 10739, 8995, 7882, 19593, 4544, 10607, 9494, 883, 3715, 5393, 15359, 13982, 3252, 9814, 7535, 12742, 9455, 7051, 17668, 18241, 8729, 11620]$
(3, 10)	$\{ \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\} \}$	0.7	36	8	360	$[0, 0, 1, 2, 3, 6, 28, 35, 24, 33, 15, 30, 22, 14, 25, 13, 17, 21, 16, 11]$
			477	10	4770	$[0, 0, 1, 3, 5, 13, 449, 466, 408, 446, 373, 427, 288, 369, 62, 343, 153, 320, 333, 125]$
			2570	12	25700	$[0, 0, 1, 3, 7, 19, 2522, 2545, 2417, 2492, 2208, 2393, 2033, 2251, 293, 2128, 867, 1963, 992, 1696]$
(3, 11)	$\{ \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\} \}$	0.73	11	6	121	$[0, 10, 10, 8, 9, 9, 8, 5, 7, 2, 6, 0, 5, 1, 4, 7, 3, 4, 2, 6, 1, 3]$
			44	8	484	$[0, 43, 43, 41, 41, 37, 40, 35, 35, 24, 33, 21, 31, 13, 30, 5, 27, 3, 19, 4, 13, 0]$
			645	10	7095	$[0, 0, 1, 3, 5, 13, 12, 29, 591, 626, 548, 606, 478, 557, 86, 533, 132, 519, 214, 396, 406, 248]$
			4000	12	44000	$[0, 0, 1, 3, 7, 19, 28, 64, 75, 170, 101, 256, 296, 556, 376, 822, 505, 1168, 1004, 2305, 1276, 3260]$
(3, 12)	$\{ \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\} \}$	0.75	13	6	156	$[0, 0, 1, 2, 2, 1, 3, 5, 4, 8, 5, 10, 7, 3, 8, 11, 6, 12, 9, 4, 10, 7, 11, 9]$
			51	8	612	$[0, 0, 1, 2, 3, 6, 4, 11, 45, 49, 41, 50, 29, 43, 27, 40, 14, 39, 19, 37, 16, 35, 9, 24]$
			837	10	10044	$[0, 0, 1, 3, 5, 13, 12, 29, 783, 818, 740, 798, 691, 773, 615, 711, 111, 683, 181, 611, 145, 522, 645, 417]$
			5100	12	61200	$[0, 0, 1, 3, 7, 19, 28, 64, 4993, 5061, 4827, 4975, 4601, 4780, 4174, 4588, 426, 4518, 729, 3952, 2231, 3725, 2217, 3834]$

Table 3: Some $\text{FSS}(v, b, 2)$ obtained from a primitive $\text{FSS}(v', b', 2)$ constructed by method I.

\mathcal{R}	(v', b')	B' (Blocks of the primitive FSS)	$2g'$	$g(B')$	(v, b)	B (Blocks of the new FSS)	$g(B)$
0.33	(2, 3)	$\{\{1, 2\}, \{1, 2\}, \{1, 2\}\}$	8	12	(6, 9)	$\{\{1, 4\}, \{2, 5\}, \{3, 6\}, \{1, 5\}, \{2, 6\}, \{3, 4\}, \{1, 6\}, \{2, 4\}, \{3, 5\}\}$	24
0.33	(2, 3)	$\{\{1, 2\}, \{1, 2\}, \{1, 2\}\}$	12	12	(14, 21)	$\{\{1, 8\}, \{2, 9\}, \{3, 10\}, \{4, 11\}, \{5, 12\}, \{6, 13\}, \{7, 14\}, \{1, 9\}, \{2, 10\}, \{3, 11\}, \{4, 12\}, \{5, 13\}, \{6, 14\}, \{7, 8\}, \{1, 11\}, \{2, 12\}, \{3, 13\}, \{4, 14\}, \{5, 8\}, \{6, 9\}, \{7, 10\}\}$	36
0.5	(2, 4)	$\{\{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 2\}\}$	8	12	(8, 16)	$\{\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}, \{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 5\}, \{1, 7\}, \{2, 8\}, \{3, 5\}, \{4, 6\}, \{1, 8\}, \{2, 5\}, \{3, 6\}, \{4, 7\}\}$	24
0.5	(2, 4)	$\{\{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 2\}\}$	12	12	(26, 52)	$\{\{1, 14\}, \{2, 15\}, \{3, 16\}, \{4, 17\}, \{5, 18\}, \{6, 19\}, \{7, 20\}, \{8, 21\}, \{9, 22\}, \{10, 23\}, \{11, 24\}, \{12, 25\}, \{13, 26\}, \{1, 15\}, \{2, 16\}, \{3, 17\}, \{4, 18\}, \{5, 19\}, \{6, 20\}, \{7, 21\}, \{8, 22\}, \{9, 23\}, \{10, 24\}, \{11, 25\}, \{12, 26\}, \{13, 14\}, \{1, 17\}, \{2, 18\}, \{3, 19\}, \{4, 20\}, \{5, 21\}, \{6, 22\}, \{7, 23\}, \{8, 24\}, \{9, 25\}, \{10, 26\}, \{11, 14\}, \{12, 15\}, \{13, 16\}, \{1, 23\}, \{2, 24\}, \{3, 25\}, \{4, 26\}, \{5, 14\}, \{6, 15\}, \{7, 16\}, \{8, 17\}, \{9, 18\}, \{10, 19\}, \{11, 20\}, \{12, 21\}, \{13, 22\}\}$	36
0.6	(2, 5)	$\{\{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 2\}\}$	8	12	(10, 25)	$\{\{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 9\}, \{5, 10\}, \{1, 7\}, \{2, 8\}, \{3, 9\}, \{4, 10\}, \{5, 6\}, \{1, 8\}, \{2, 9\}, \{3, 10\}, \{4, 6\}, \{5, 7\}, \{1, 9\}, \{2, 10\}, \{3, 6\}, \{4, 7\}, \{5, 8\}, \{1, 10\}, \{2, 6\}, \{3, 7\}, \{4, 8\}, \{5, 9\}\}$	24
0.6	(2, 5)	$\{\{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 2\}\}$	12	12	(46, 115)	$\{\{1, 24\}, \{2, 25\}, \{3, 26\}, \{4, 27\}, \{5, 28\}, \{6, 29\}, \{7, 30\}, \{8, 31\}, \{16, 39\}, \{17, 40\}, \{18, 41\}, \{19, 42\}, \{20, 43\}, \{21, 44\}, \{22, 45\}, \{23, 46\}, \{1, 25\}, \{2, 26\}, \{3, 27\}, \{4, 28\}, \{5, 29\}, \{6, 30\}, \{7, 31\}, \{8, 32\}, \{9, 33\}, \{10, 34\}, \{11, 35\}, \{12, 36\}, \{13, 37\}, \{14, 38\}, \{15, 39\}, \{16, 40\}, \{17, 41\}, \{18, 42\}, \{19, 43\}, \{20, 44\}, \{21, 45\}, \{22, 46\}, \{23, 24\}, \{1, 27\}, \{2, 28\}, \{3, 29\}, \{4, 30\}, \{5, 31\}, \{6, 32\}, \{7, 33\}, \{8, 34\}, \{9, 35\}, \{10, 36\}, \{11, 37\}, \{12, 38\}, \{13, 39\}, \{14, 40\}, \{15, 41\}, \{16, 42\}, \{17, 43\}, \{18, 44\}, \{19, 45\}, \{20, 46\}, \{21, 24\}, \{22, 25\}, \{23, 26\}, \{1, 32\}, \{2, 33\}, \{3, 34\}, \{4, 35\}, \{5, 36\}, \{6, 37\}, \{7, 38\}, \{8, 39\}, \{9, 40\}, \{10, 41\}, \{11, 42\}, \{12, 43\}, \{13, 44\}, \{14, 45\}, \{15, 46\}, \{16, 24\}, \{17, 25\}, \{18, 26\}, \{19, 27\}, \{20, 28\}, \{21, 29\}, \{22, 30\}, \{23, 31\}, \{1, 38\}, \{2, 39\}, \{3, 40\}, \{4, 41\}, \{5, 42\}, \{6, 43\}, \{7, 44\}, \{8, 45\}, \{9, 46\}, \{10, 24\}, \{11, 25\}, \{12, 26\}, \{13, 27\}, \{14, 28\}, \{15, 29\}, \{16, 30\}, \{17, 31\}, \{18, 32\}, \{19, 33\}, \{20, 34\}, \{21, 35\}, \{22, 36\}, \{23, 37\}\}$	36
0.66	(2, 6)	$\{\{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 2\}\}$	8	12	(12, 36)	$\{\{1, 7\}, \{2, 8\}, \{3, 9\}, \{4, 10\}, \{5, 11\}, \{6, 12\}, \{1, 8\}, \{2, 9\}, \{3, 10\}, \{4, 11\}, \{5, 12\}, \{6, 7\}, \{1, 9\}, \{2, 10\}, \{3, 11\}, \{4, 12\}, \{5, 7\}, \{6, 8\}, \{1, 10\}, \{2, 11\}, \{3, 12\}, \{4, 7\}, \{5, 8\}, \{6, 9\}, \{1, 11\}, \{2, 12\}, \{3, 7\}, \{4, 8\}, \{5, 9\}, \{6, 10\}, \{1, 12\}, \{2, 7\}, \{3, 8\}, \{4, 9\}, \{5, 10\}, \{6, 11\}\}$	24
0.25	(3, 4)	$\{\{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}\}$	6	12	(15, 20)	$\{\{1, 6, 11\}, \{2, 7, 12\}, \{3, 8, 13\}, \{4, 9, 14\}, \{5, 10, 15\}, \{1, 7, 13\}, \{2, 8, 14\}, \{3, 9, 15\}, \{4, 10, 11\}, \{5, 6, 12\}, \{1, 8, 15\}, \{2, 9, 11\}, \{3, 10, 12\}, \{4, 6, 13\}, \{5, 7, 14\}, \{1, 9, 12\}, \{2, 10, 13\}, \{3, 6, 14\}, \{4, 7, 15\}, \{5, 8, 11\}\}$	18
0.4	(3, 5)	$\{\{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}\}$	6	12	(15, 25)	$\{\{1, 6, 11\}, \{2, 7, 12\}, \{3, 8, 13\}, \{4, 9, 14\}, \{5, 10, 15\}, \{1, 7, 13\}, \{2, 8, 14\}, \{3, 9, 15\}, \{4, 10, 11\}, \{5, 6, 12\}, \{1, 8, 15\}, \{2, 9, 11\}, \{3, 10, 12\}, \{4, 6, 13\}, \{5, 7, 14\}, \{1, 9, 12\}, \{2, 10, 13\}, \{3, 6, 14\}, \{4, 7, 15\}, \{5, 8, 11\}, \{1, 10, 14\}, \{2, 6, 15\}, \{3, 7, 11\}, \{4, 8, 12\}, \{5, 9, 13\}\}$	18
0.4	(3, 5)	$\{\{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}\}$	8	12	(42, 70)	$\{\{1, 15, 29\}, \{2, 16, 30\}, \{3, 17, 31\}, \{4, 18, 32\}, \{5, 19, 33\}, \{6, 20, 34\}, \{7, 21, 35\}, \{8, 22, 36\}, \{9, 23, 37\}, \{10, 24, 38\}, \{11, 25, 39\}, \{12, 26, 40\}, \{13, 27, 41\}, \{14, 28, 42\}, \{1, 16, 31\}, \{2, 17, 32\}, \{3, 18, 33\}, \{4, 19, 34\}, \{5, 20, 35\}, \{6, 21, 36\}, \{7, 22, 37\}, \{8, 23, 38\}, \{9, 24, 39\}, \{10, 25, 40\}, \{11, 26, 41\}, \{12, 27, 42\}, \{13, 28, 29\}, \{14, 15, 30\}, \{1, 18, 36\}, \{2, 19, 37\}, \{3, 20, 38\}, \{4, 21, 39\}, \{5, 22, 40\}, \{6, 23, 41\}, \{7, 24, 42\}, \{8, 25, 29\}, \{9, 26, 30\}, \{10, 27, 31\}, \{11, 28, 32\}, \{12, 15, 33\}, \{13, 16, 34\}, \{14, 17, 35\}, \{1, 19, 41\}, \{2, 20, 42\}, \{3, 21, 29\}, \{4, 22, 30\}, \{5, 23, 31\}, \{6, 24, 32\}, \{7, 25, 33\}, \{8, 26, 34\}, \{9, 27, 35\}, \{10, 28, 36\}, \{11, 15, 37\}, \{12, 16, 38\}, \{13, 17, 39\}, \{14, 18, 40\}, \{1, 20, 39\}, \{2, 21, 40\}, \{3, 22, 41\}, \{4, 23, 42\}, \{5, 24, 29\}, \{6, 25, 30\}, \{7, 26, 31\}, \{8, 27, 32\}, \{9, 28, 33\}, \{10, 15, 34\}, \{11, 16, 35\}, \{12, 17, 36\}, \{13, 18, 37\}, \{14, 19, 38\}\}$	24
0.4	(3, 6)	$\{\{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}\}$	6	12	(21, 42)	$\{\{1, 8, 15\}, \{2, 9, 16\}, \{3, 10, 17\}, \{4, 11, 18\}, \{5, 12, 19\}, \{6, 13, 20\}, \{7, 14, 21\}, \{1, 9, 17\}, \{2, 10, 18\}, \{3, 11, 19\}, \{4, 12, 20\}, \{5, 13, 21\}, \{6, 14, 15\}, \{7, 8, 16\}, \{1, 10, 19\}, \{2, 11, 20\}, \{3, 12, 21\}, \{4, 13, 15\}, \{5, 14, 16\}, \{6, 8, 17\}, \{7, 9, 18\}, \{1, 11, 21\}, \{2, 12, 15\}, \{3, 13, 16\}, \{4, 14, 17\}, \{5, 8, 18\}, \{6, 9, 19\}, \{7, 10, 20\}, \{1, 12, 16\}, \{2, 13, 17\}, \{3, 14, 18\}, \{4, 8, 19\}, \{5, 9, 20\}, \{6, 10, 21\}, \{7, 11, 15\}, \{1, 13, 18\}, \{2, 14, 19\}, \{3, 8, 20\}, \{4, 9, 21\}, \{5, 10, 15\}, \{6, 11, 16\}, \{7, 12, 17\}\}$	18
0.33	(4, 6)	$\{\{1, 2, 3, 4\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4\}\}$	6	12	(28, 42)	$\{\{1, 8, 15, 22\}, \{2, 9, 16, 23\}, \{3, 10, 17, 24\}, \{4, 11, 18, 25\}, \{5, 12, 19, 26\}, \{6, 13, 20, 27\}, \{7, 14, 21, 28\}, \{1, 9, 17, 25\}, \{2, 10, 18, 26\}, \{3, 11, 19, 27\}, \{4, 12, 20, 28\}, \{5, 13, 21, 22\}, \{6, 14, 15, 23\}, \{7, 8, 16, 24\}, \{1, 10, 19, 28\}, \{2, 11, 20, 22\}, \{3, 12, 21, 23\}, \{4, 13, 15, 24\}, \{5, 14, 16, 25\}, \{6, 8, 17, 26\}, \{7, 9, 18, 27\}, \{1, 11, 21, 24\}, \{2, 12, 15, 25\}, \{3, 13, 16, 26\}, \{4, 14, 17, 27\}, \{5, 8, 18, 28\}, \{6, 9, 19, 22\}, \{7, 10, 20, 23\}, \{1, 12, 16, 27\}, \{2, 13, 17, 28\}, \{3, 14, 18, 22\}, \{4, 8, 19, 23\}, \{5, 9, 20, 24\}, \{6, 10, 21, 25\}, \{7, 11, 15, 26\}, \{1, 13, 18, 23\}, \{3, 14, 18, 22\}, \{4, 8, 19, 23\}, \{5, 9, 20, 24\}, \{6, 10, 21, 25\}, \{7, 11, 15, 26\}, \{1, 13, 18, 23\}, \{2, 14, 19, 24\}, \{3, 8, 20, 25\}, \{4, 9, 21, 26\}, \{5, 10, 15, 27\}, \{6, 11, 16, 28\}, \{7, 12, 17, 22\}\}$	18

Table 4: Some new $\text{FSS}(v, b, 2)$ obtained from a primitive $\text{FSS}(v', b', 2)$ constructed by method II.

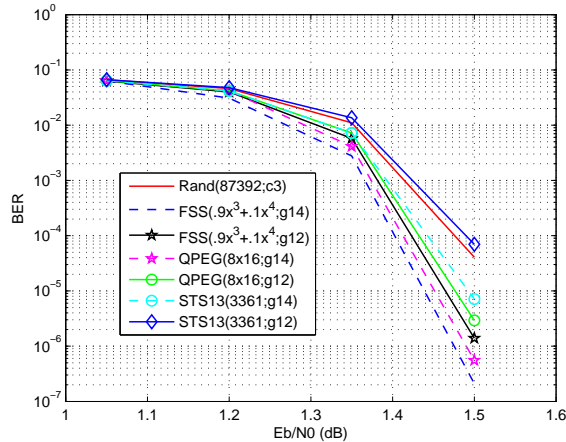
[illegible]

Figure 4: A comparison between some FSS codes constructed by method II, QPEG, STS(13) and random-like LDPC codes of length 87000 and rate about 0.5.

Table 5: New column-weight two FSS($v, b, 2$) of rate \mathcal{R} against the best previous found rate r .

(v, b)	$g(\mathcal{B})$	\mathcal{R}	r	\mathcal{B}
(7, 11)	24	0.36	0.33 [17]	$\{[1, 2], \{4, 6\}, \{1, 6\}, \{2, 7\}, \{4, 7\}, \{1, 3\}, \{4, 7\}, \{5, 7\}, \{3, 5\}, \{1, 6\}, \{3, 5\}]\}$
(8, 13)	24	0.38		$\{[1, 2], \{2, 3\}, \{2, 8\}, \{6, 7\}, \{1, 4\}, \{1, 3\}, \{5, 8\}, \{7, 8\}, \{5, 6\}, \{1, 6\}, \{4, 7\}, \{4, 8\}, \{2, 5\}]\}$
(9, 16)	24	0.44		$\{[1, 2], \{3, 6\}, \{3, 6\}, \{8, 9\}, \{5, 7\}, \{7, 9\}, \{5, 6\}, \{7, 9\}, \{4, 7\}, \{1, 9\}, \{2, 8\}, \{3, 8\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{4, 6\}]\}$
(12, 26)	24	0.54		$\{[1, 2], \{8, 12\}, \{3, 8\}, \{3, 4\}, \{9, 10\}, \{6, 12\}, \{7, 11\}, \{2, 12\}, \{8, 9\}, \{6, 12\}, \{3, 8\}, \{10, 11\}, \{11, 12\}, \{4, 11\}, \{2, 7\}, \{7, 8\}, \{9, 11\}, \{1, 10\}, \{5, 6\}, \{5, 7\}, \{2, 9\}, \{4, 5\}, \{2, 4\}, \{1, 5\}, \{5, 10\}, \{1, 8\}]\}$
(14, 33)	24	0.58	0.25 [20]	$\{[1, 2], \{1, 9\}, \{9, 11\}, \{13, 14\}, \{12, 13\}, \{1, 14\}, \{12, 13\}, \{5, 11\}, \{6, 12\}, \{8, 14\}, \{4, 9\}, \{4, 5\}, \{11, 14\}, \{4, 6\}, \{10, 12\}, \{10, 11\}, \{9, 13\}, \{5, 6\}, \{7, 14\}, \{2, 13\}, \{6, 8\}, \{2, 5\}, \{1, 6\}, \{4, 10\}, \{3, 7\}, \{1, 10\}, \{8, 11\}, \{5, 7\}, \{3, 11\}, \{3, 9\}, \{7, 10\}, \{3, 6\}, \{2, 8\}]\}$
(14, 21)	32	0.33		$\{[1, 2], \{2, 10\}, \{12, 13\}, \{11, 14\}, \{6, 9\}, \{6, 12\}, \{4, 12\}, \{5, 10\}, \{3, 14\}, \{6, 8\}, \{4, 14\}, \{8, 12\}, \{1, 11\}, \{10, 13\}, \{1, 7\}, \{5, 9\}, \{7, 8\}, \{2, 4\}, \{9, 11\}, \{3, 5\}, \{3, 7\}]\}$
(15, 25)	32	0.4		$\{[3, 12], \{12, 13\}, \{14, 15\}, \{4, 12\}, \{5, 7\}, \{1, 3\}, \{4, 7\}, \{13, 14\}, \{9, 10\}, \{10, 15\}, \{6, 14\}, \{2, 8\}, \{10, 11\}, \{1, 2\}, \{4, 6\}, \{5, 15\}, \{5, 13\}, \{1, 14\}, \{5, 8\}, \{3, 9\}, \{6, 9\}, \{7, 11\}, \{2, 4\}, \{3, 11\}, \{8, 9\}]\}$
(16, 27)	32	0.41		$\{[3, 13], \{4, 11\}, \{9, 11\}, \{9, 12\}, \{2, 11\}, \{6, 14\}, \{6, 9\}, \{4, 16\}, \{7, 14\}, \{3, 10\}, \{4, 5\}, \{5, 15\}, \{13, 16\}, \{8, 12\}, \{9, 10\}, \{12, 13\}, \{1, 14\}, \{8, 11\}, \{10, 15\}, \{1, 12\}, \{6, 16\}, \{7, 8\}, \{2, 3\}, \{2, 14\}, \{1, 15\}, \{5, 7\}, \{15, 16\}]\}$
(17, 30)	32	0.43	0.2 [20]	$\{[10, 15], \{16, 17\}, \{4, 5\}, \{2, 7\}, \{8, 11\}, \{6, 8\}, \{1, 4\}, \{10, 13\}, \{8, 9\}, \{5, 16\}, \{1, 2\}, \{4, 14\}, \{5, 13\}, \{7, 9\}, \{1, 15\}, \{6, 17\}, \{10, 14\}, \{11, 16\}, \{1, 6\}, \{2, 13\}, \{11, 14\}, \{7, 14\}, \{3, 4\}, \{15, 16\}, \{12, 13\}, \{11, 12\}, \{7, 17\}, \{5, 9\}, \{3, 12\}, \{3, 6\}]\}$
(19, 34)	32	0.44		$\{[4, 10], \{18, 19\}, \{2, 5\}, \{4, 11\}, \{9, 19\}, \{2, 16\}, \{4, 6\}, \{7, 15\}, \{14, 16\}, \{6, 17\}, \{8, 19\}, \{3, 7\}, \{1, 3\}, \{15, 16\}, \{5, 10\}, \{1, 11\}, \{11, 19\}, \{3, 10\}, \{7, 18\}, \{2, 19\}, \{6, 18\}, \{6, 14\}, \{9, 15\}, \{8, 17\}, \{3, 13\}, \{7, 12\}, \{12, 13\}, \{12, 14\}, \{3, 8\}, \{2, 13\}, \{5, 17\}, \{15, 17\}, \{1, 16\}, \{9, 10\}]\}$
(17, 23)	40	0.26	0.17 [20]	$\{[6, 11], \{5, 12\}, \{4, 14\}, \{13, 16\}, \{5, 8\}, \{4, 7\}, \{15, 16\}, \{13, 17\}, \{5, 17\}, \{9, 14\}, \{2, 10\}, \{7, 15\}, \{11, 12\}, \{2, 8\}, \{1, 4\}, \{3, 9\}, \{3, 16\}, \{10, 13\}, \{3, 6\}, \{2, 7\}, \{9, 17\}, \{1, 8\}, \{1, 6\}]\}$
(20, 28)	40	0.29		$\{[4, 16], \{6, 10\}, \{11, 14\}, \{7, 18\}, \{1, 14\}, \{15, 16\}, \{11, 15\}, \{3, 5\}, \{1, 13\}, \{13, 18\}, \{11, 12\}, \{9, 19\}, \{6, 15\}, \{8, 10\}, \{4, 20\}, \{1, 8\}, \{17, 20\}, \{9, 12\}, \{4, 7\}, \{5, 17\}, \{5, 8\}, \{19, 20\}, \{3, 18\}, \{7, 11\}, \{13, 19\}, \{2, 6\}, \{2, 17\}, \{5, 12\}]\}$
(22, 31)	40	0.29		$\{[1, 2], \{12, 19\}, \{11, 21\}, \{9, 14\}, \{20, 22\}, \{14, 15\}, \{9, 20\}, \{8, 13\}, \{21, 22\}, \{5, 12\}, \{5, 14\}, \{8, 18\}, \{6, 15\}, \{14, 21\}, \{1, 7\}, \{10, 11\}, \{13, 20\}, \{3, 10\}, \{16, 19\}, \{17, 19\}, \{1, 21\}, \{4, 8\}, \{4, 17\}, \{2, 3\}, \{16, 18\}, \{8, 10\}, \{2, 19\}, \{3, 6\}, \{15, 17\}, \{16, 22\}, \{4, 7\}]\}$
(23, 33)	40	0.3		$\{[12, 13], \{12, 15\}, \{11, 21\}, \{4, 22\}, \{6, 10\}, \{3, 4\}, \{21, 23\}, \{8, 14\}, \{8, 20\}, \{5, 15\}, \{2, 7\}, \{19, 21\}, \{16, 18\}, \{6, 17\}, \{1, 15\}, \{2, 14\}, \{5, 10\}, \{6, 18\}, \{13, 19\}, \{7, 18\}, \{13, 14\}, \{4, 17\}, \{9, 12\}, \{16, 22\}, \{7, 23\}, \{16, 20\}, \{20, 21\}, \{1, 11\}, \{2, 3\}, \{10, 11\}, \{17, 19\}, \{3, 5\}, \{9, 17\}]\}$
(40, 62)	40	0.35	0.2 [20]	$\{[26, 29], \{26, 36\}, \{27, 29\}, \{8, 28\}, \{39, 40\}, \{4, 17\}, \{19, 29\}, \{15, 24\}, \{12, 23\}, \{25, 32\}, \{8, 17\}, \{26, 28\}, \{17, 23\}, \{21, 30\}, \{4, 23\}, \{2, 25\}, \{16, 18\}, \{11, 16\}, \{11, 15\}, \{27, 33\}, \{32, 35\}, \{3, 20\}, \{1, 6\}, \{2, 38\}, \{20, 37\}, \{3, 5\}, \{35, 37\}, \{24, 32\}, \{9, 31\}, \{9, 30\}, \{30, 34\}, \{3, 6\}, \{5, 38\}, \{19, 28\}, \{33, 37\}, \{5, 22\}, \{24, 26\}, \{22, 32\}, \{13, 15\}, \{18, 36\}, \{14, 24\}, \{16, 33\}, \{9, 13\}, \{2, 12\}, \{7, 18\}, \{10, 14\}, \{37, 39\}, \{5, 13\}, \{1, 27\}, \{28, 31\}, \{1, 34\}, \{20, 31\}, \{9, 10\}, \{7, 34\}, \{6, 11\}, \{36, 38\}, \{18, 35\}, \{30, 39\}, \{12, 14\}, \{25, 34\}, \{19, 40\}, \{23, 33\}]\}$
(26, 32)	48	0.19		$\{[21, 22], \{7, 11\}, \{19, 21\}, \{15, 24\}, \{12, 25\}, \{11, 15\}, \{25, 26\}, \{23, 24\}, \{11, 15\}, \{8, 25\}, \{6, 26\}, \{2, 23\}, \{9, 18\}, \{4, 23\}, \{4, 10\}, \{4, 8\}, \{10, 18\}, \{3, 8\}, \{14, 21\}, \{5, 21\}, \{22, 26\}, \{2, 13\}, \{16, 19\}, \{9, 14\}, \{1, 5\}, \{13, 14\}, \{10, 12\}, \{1, 20\}, \{17, 19\}, \{20, 26\}, \{7, 17\}, \{16, 18\}]\}$
(27, 35)	48	0.23	0.17 [20]	$\{[1, 20], \{21, 23\}, \{14, 18\}, \{14, 22\}, \{9, 10\}, \{6, 20\}, \{12, 25\}, \{16, 20\}, \{24, 26\}, \{13, 25\}, \{5, 24\}, \{11, 18\}, \{4, 8\}, \{4, 22\}, \{12, 21\}, \{16, 22\}, \{15, 24\}, \{10, 19\}, \{4, 27\}, \{11, 25\}, \{19, 23\}, \{2, 23\}, \{17, 21\}, \{8, 12\}, \{1, 15\}, \{3, 8\}, \{2, 26\}, \{26, 27\}, \{15, 18\}, \{9, 13\}, \{6, 13\}, \{7, 14\}, \{3, 5\}, \{16, 17\}, \{7, 10\}]\}$
(28, 37)	48	0.24		$\{[1, 2], \{18, 22\}, \{25, 28\}, \{16, 24\}, \{21, 28\}, \{6, 8\}, \{10, 16\}, \{6, 11\}, \{13, 21\}, \{1, 18\}, \{16, 19\}, \{7, 23\}, \{1, 27\}, \{23, 26\}, \{6, 24\}, \{3, 4\}, \{3, 9\}, \{3, 7\}, \{5, 10\}, \{20, 24\}, \{17, 26\}, \{9, 19\}, \{2, 28\}, \{11, 13\}, \{8, 22\}, \{19, 26\}, \{2, 4\}, \{7, 12\}, \{8, 12\}, \{2, 14\}, \{14, 20\}, \{15, 23\}, \{11, 17\}, \{22, 25\}, \{16, 27\}, \{15, 18\}, \{5, 21\}]\}$
(29, 39)	48	0.26		$\{[1, 2], \{6, 11\}, \{22, 23\}, \{18, 21\}, \{2, 27\}, \{26, 28\}, \{23, 28\}, \{13, 20\}, \{23, 24\}, \{1, 14\}, \{6, 12\}, \{1, 4\}, \{3, 15\}, \{1, 10\}, \{13, 27\}, \{5, 29\}, \{18, 19\}, \{5, 7\}, \{3, 12\}, \{7, 19\}, \{11, 16\}, \{24, 25\}, \{17, 28\}, \{9, 22\}, \{12, 29\}, \{26, 29\}, \{18, 20\}, \{3, 9\}, \{6, 20\}, \{2, 7\}, \{10, 15\}, \{14, 22\}, \{11, 25\}, \{16, 17\}, \{24, 27\}, \{4, 16\}, \{8, 19\}, \{15, 21\}, \{17, 19\}]\}$

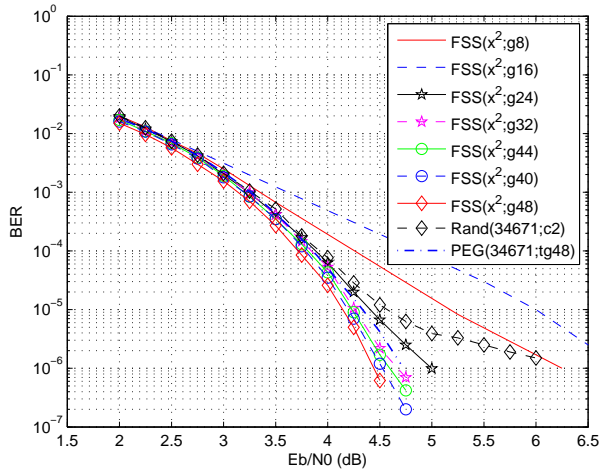


Figure 5: A comparison between some column-weight two FSS codes with different girths, random-like and PEG LDPC codes of length 34671 and rate about 0.26.

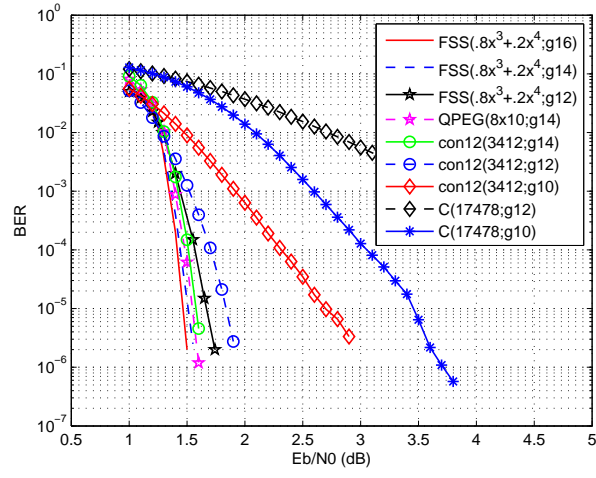


Figure 6: A comparison between some non-regular FSS codes, QPEG, Aff*(16)-based and cylinder-type QC LDPC codes of length 58000 and rate about 0.2.